Analogies Between Boolean Algebra, Set Theory, and Real Function Spaces

Luciano da Fontoura Costa
luciano@ifsc.usp.br

São Carlos Institute of Physics – DFCM/USP

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Abstract

This work addresses the possibility of associating concepts, structures and operations among Boolean algebra, set theory, and real functions. For instance, logical negation becomes set complement, and then sign change in real function spaces. Similarly, the logical and operation becomes set intersection, and then the minimum binary operator. While these analogies are, in principle, only conjectural, they can nevertheless be eventually proved in the respective domains. Thus, the proposed approach here can be understood as establishing bridges of analogies between the three considered domains that can provide insights to be subsequently validated (or not) within those domains. It is also conjectured that there exists a unifying correspondance between all the analogous structures, operations and properties in these three domains, including all the 16 possible logical operations between two Boolean variables. Examples of these interesting possibilities are also presented in this work, including the verification of the De Morgan theorem for real functions and a comprehensive characterization of the recently introduced common product, with several important implications for areas including pattern recognition, deep learning, signal processing, and scientific modeling.

'The firmament on a calm lake, united into a single plane and then reflected caleidoscopically into my window glasses.'

LdaFC

1 Introduction

Functions in real spaces can be studied, characterized and operated in a virtually infinite number of possibilities. Among the several approaches that can be used for that finality, the following three approaches are particularly important and frequently used, being characterized by the consideration of: (i) individual functions; (ii) functionals defined on these functions; (iii) binary operators on functions; (iv) functionals on binary operators. By binary operator is meant functions that two functions are taken as argument, yielding a third as a result. There are several other possible approaches, but several of them can be handled and understood in terms of the four cases above.

Developments in mathematics and the physical sciences have largely relied on basic algebraic operations that include functions such as harmonic, polynomial and binary operations of sum, product, and division. Though the operations of maximum, minimum and sign of a function are well defined and robust in real function spaces, they do not seem to have been employed as often as the aforementioned algebraic functions and operations.

Based on recent results [3, 1, 2], the present work elaborates on the suggested possibility [3] of developing an analogy between Boolean algebra, set theory, and real function spaces. For instance, the algebraic operation $min(f, g)$, where f and g are any two real functions with a common support, becomes understood as the '*intersection*' $f \cap g$ between those functions, $-f$ as the '*complement*' f^C of function f, and the integral of a function over a specified support as its respective '*cardinality*'. These analogies will mostly involve the *sign function* and the operations of *maximum* and *minimum*.

Table 1, extracted from [2], illustrates some of the many possibly analogies that can be considered between the domains of Boolean algebra, set theory, and real function spaces.

Observe that these analogies are established only in a conceptual manner, requiring no changes whatsoever in any of the three considered spaces. The point here is that the relationships between the three considered domains, with emphasis on the 16 possible Boolean operations between two logic variable, can lead to hypothesis which can

logical	$\neg$	$\wedge$	$\vee$
set theoretical	$[]^C$	$\cap$	$\cup$
algebraic	$-$	$*$	$+$

Table 1: The potential interrelationships between three basic operations at the logic, set, and algebraic levels (from [3]).

be subsequently validated in the respective domains.

In particular, the present work proposes the conjecture that all the analogies between the three considered domains are valid.

For simplicity's sake, we adopt set notation for dealing with functions, but it should be kept in mind that we are just meaning conceptual analogies.

Though motivated with the areas of scientific modeling, complexity, pattern recognition and signal processing in mind, the reported developments can also be further explored from a more theoretical point of view.

We start by describing the main associations between set and function theories, motivated by recently described results [1, 2], and then proceed to addressing the four possibilities indicated above.

2 Single Functions and Functionals

Each possible single real function can be understood as corresponding to a '*set*'. For instance, the real function $f(x)$ becomes f. In case the independent variable x needs to be identified, we can have f_x.

Given a function $f(x)$, $-f(x)$ is conceptually understood as the *complement* f^C of f.

The domain of a function $f(x)$ becomes associated to its *support*.

A function of particular interest is the *sign* of a function, which is here expressed as:

$$sgn(f(x)) = s_f = \begin{cases} -1 \iff f(x) < 0 \\ 0 \iff f(x) = 0 \\ 1 \iff f(x) > 0 \end{cases} \tag{1}$$

The absolute value can then be expressed as:

$$|f(x)| = s_f f \tag{2}$$

The integral of a function f is henceforth conceptually understood as its *cardinality*:

$$\backslash f \backslash = \int_{-\infty}^{\infty} f(x) dx \tag{3}$$

We can now write respectively to the cardinality:

$$-f = s_f f = f^C = -f \tag{4}$$

$$\backslash - f \backslash = \backslash s_f f \backslash = \backslash f^C \backslash = -\backslash f \backslash \tag{5}$$

and, with respect to the sign function:

$$(s_f)(s_f) = 1 \tag{6}$$

$$(s_f)^n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases} \tag{7}$$

3 Binary Operations

Though there are 16 possible logical operations between two Boolean variables (e.g. [3]), in the present work we will shall focus on the operations of union, intersection, sum, and subtraction. The other 12 possible binary operations will be addressed in subsequent works.

It is henceforth assumed that all pairs of functions considered by the binary operators share their respective supports.

The *union* between two functions f and g corresponds to:

$$f \cup g = \min \{f, g\} \tag{8}$$

The *intersection* between two functions f and g corresponds to:

$$f \cap g = \min \{f, g\} \tag{9}$$

The *sum* and *subtraction* between two functions f and g are associated respectively to $f - g$ and $f + g$.

It is interesting to consider how the sign function in the context of binary operators. In particular, we define the conjoint sign function as:

$$s_{f,g} = (s_f)(s_g) \tag{10}$$

The analogy of the De Morgan theorem in real function spaces can now be expressed as:

$$[f \cup g]^C = f^C \cap g^C \tag{11}$$

$$[f \cap g]^C = f^C \cup g^C \tag{12}$$

In order to verify the validity of this putative property, we translate back the above equations into the real function notation as:

$$-\min \{f(x), g(x)\} = \max \{-f(x), -g(x)\} \tag{13}$$

$$-\max \{f(x), g(x)\} = \min \{-f(x), -g(x)\} \tag{14}$$

which is evidently satisfied. This development provides an example of how insights derived from set theory, as well as the respectively associated logical (or Boolean) operations [3], can be verified for real functions.

Indeed, let's also consider the real function analogy of the exclusive or logical operation translated to sets, i.e.:

$$A \oplus B = (\neg A \wedge B) \cup (A \wedge \neg B) \tag{15}$$

The *exclusive or* can now be associated to two real functions f and g as:

$$f \oplus g = (f^C \cap g) \cup (f \cap g^C) \tag{16}$$

which can be immediately expanded as:

$$\max\{\min\{-f(x), g(x)\}, \min\{f(x), -g(x)\}\} \quad (17)$$

therefore also corresponding to a valid operation in the space of real functions, with several related properties.

Observe that, in this case, we are considering two successive associations, one from logical operations into sets, and then from sets to real functions.

4 Binary Operation Functionals

The possibly most important functional in real function spaces is the *inner product*, between two functions $f(x)$ and $g(x)$, which is defined as:

$$\langle f(x), g(x)\rangle = \int_S f(x)g(x)dx \quad (18)$$

where S is a valid considered domain (or support).

The inner product has special conceptual interpretation as providing a possible quantification of the similarity between the two functions. This becomes more evident respectively to the inner product of two vectors, which is proportional to the cosine of the smallest angle between the vectors.

A possible analogue of the inner product in multisets has been proposed [2], corresponding to:

$$\ll f(x), g(x) \gg = \int_S s_{f,g} \min\{s_f f(x), s_g g(x)\} dx \quad (19)$$

or, in the notation described in this work:

$$\ll f, g \gg = \backslash s_{f,g}\left[s_f f \cap s_g g\right] \backslash \quad (20)$$

This concept, which corresponds to a valid functional totally immersed in the real function space, irrespectively to sets or multisets, has been called the *common product* between the two functions $f(x)$ and $g(x)$.

It is then possible to derive a sliding functional operation analogous to the cross-correlation in real function spaces as:

$$f(x) \diamond g(x)[y] = \int_S \ll f(x)g(x-y) \gg dx \quad (21)$$

which translates into:

$$\ll f, g \gg [y] \;= \backslash s_{f,g}\left[s_f f \cap s_g g_{x-y}\right] \backslash \, [y] \quad (22)$$

an operation that will here be called the *cosimilarity* between the functions $f(x)$ and $g(x)$.

Similarly a sliding operation analogous to the convolution between two real functions corresponds to:

$$f(x) \Box g(x)[y] = \int_S \ll f(x)g(y-x) \gg dx \quad (23)$$

which translates into:

$$f \Box g[y] \;= \backslash s_{f,g}\left[s_f f \cap s_g g_{y-x}\right] \backslash \, [y] \quad (24)$$

Now, it is interesting to consider the functional

$$f(x) \circ g(x) = s_{f,g} \min(s_f f(x), s_g g(x)) \quad (25)$$

which is associated to:

$$f \circ g = s_{f,g}\left[(s_f f) \cap (s_g g)\right] \quad (26)$$

The binary operator $f \circ g$ is henceforth referred to as *sproduct*, appearing in all the above functionals and respectively derived sliding functionals. As such, this product deserves further analysis, which is done as follows.

Let's consider the specific case of the sine and cosine functions, illustrated in Figure 1 jointly with their respective sign functions.

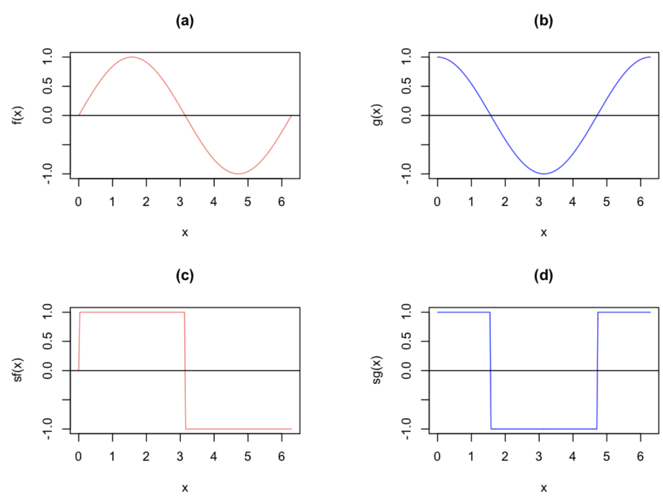

Figure 1: The sine (a) and cosine (b) functions, as well as their respective sign functions (c) and (d).

The function $(s_f f) \cap (s_g g)$ is obtained by multiplying the previous function by the conjoint sign function $s_{f,g}$, being shown in Figure 2.

The sproduct of the sine and cosine, i.e. $f \circ g$, is depicted in Figure 3 jointly with the function $s_{f,g}$.

Therefore, it becomes clear that the function $(s_f f) \cap (s_g g)$ identifies the common areas (hence the name *common product* assigned to the respective functional) between the functions $s_f f$ and $s_g g$. This can be understood as a kind of set intersection that takes into account the signal of the functions so as to consider the common area comprised between the functions and the horizontal axis.

The subsequent product by $s_{f,g}$ then associates signs to the common areas.

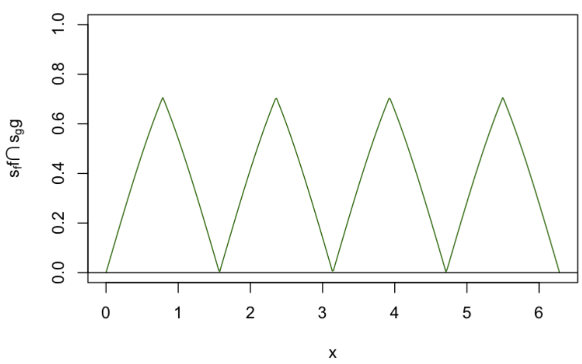

Figure 2: The function $(s_f f) \cap (s_g g)$.

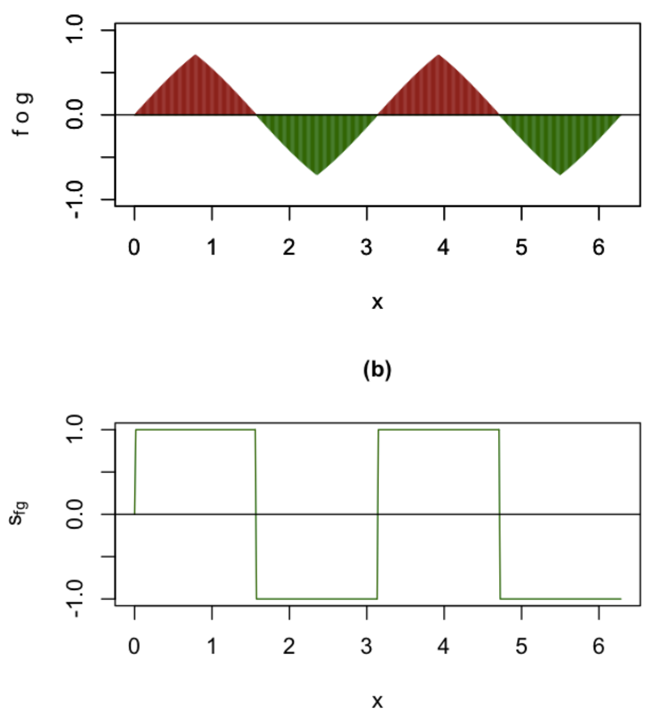

Figure 3: The sproduct $f \circ g$ of the sine and cosine functions (a) and the conjoint sign function $s_{f,g}(x)$ (b). The common product for these two functions is zero.

Now, it becomes clear that the cardinality of $f \circ g$, which corresponds to the integral of $s_{f,g}$ along the specified support S, actually quantifies the net similarity between the two functions f and g in directly analogy with inner product action with respect to the algebraic product between two functions. In the case of the sine and cosine, the common product is readily verified to be zero.

Figure 4 presents two other functions f and g.

The respectively obtained sproduct is shown in Figure 5, together with the conjoint sign function $s_{f,g}$. The resulting common product in this case is equal to 29.68.

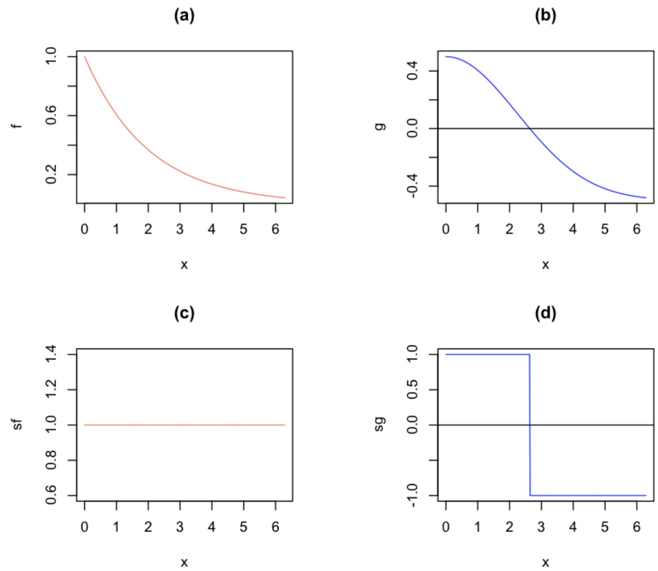

Figure 4: Two functions $f(x) = e^{-0.5x}$ (a) and $g(x) = e^{-0.1x^2}$ (b), and their respective sign functions (c) and (d).

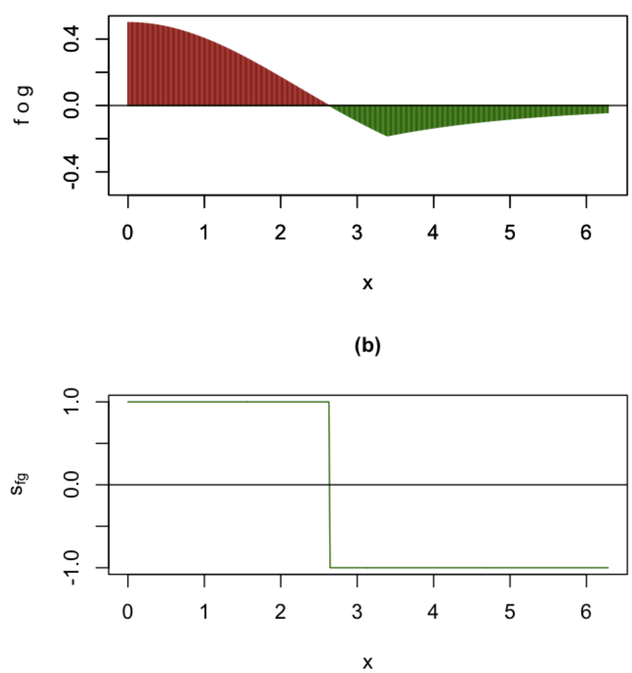

Figure 5: The sproduct $f \circ g$ of the sine and cosine functions (a) and the function $s_{f,g}(x)$ (b). The common product in this case is 29.68.

5 Concluding Remarks

Boolean variables, sets and real function spaces are assuredly quite different mathematical structures. Yet, there are intrinsic analogies between these three domains that can be considered in a more systematic manner so as to provide cross-related insights and bridges between

structures and properties in these three realms [3].

The present work presented a preliminary exploration of these possibilities. More specifically, real functions were associated to set theoretical constructs and properties motivated by the recent approaches [1, 2].

The reported development provided several examples of interesting results that can be obtained, including the verification of the De Morgan theorem (originating from Boolean algebra) for real function spaces. In addition, we also developed a study of the recently introduced concept of common product while considering concepts from both real function spaces as well as set theory analogies.

A large number of related developments are possible, including the systematic identification of the valid analogies between all the 16 possible Boolean algebra operations with set theory and real function spaces.

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